Differential Geometry IV: General Relativity

7.1 For the following spacetimes, decide if they are globally hyperbolic or not. If they are, find a Cauchy hypersurface.

(a) The future timecone

$$I_+[0] = \{(x^0, \dots, x^n) : -(x^0)^2 + (x^1)^2 + \dots + (x^n)^2 < 0 \quad \text{and} \quad x^0 > 0\}$$

inside Minkowski spacetime (equipped with the Minkowski metric η).

(b) The spacetime ($\mathbb{R}^{1+3}$) equipped with a Lorentzian metric g which satisfies, in the Cartesian coordinates,

$$|g_{\alpha\beta} - \eta_{\alpha\beta}| < \frac{1}{10}$$

(c) The $1+1$ dimensional Anti-de Sitter spacetime from Exercise 6.3.

7.2 Let $(\mathcal{M}, g)$ be a spacetime and let $p \in \mathcal{M}$.

(a) Show that if $q \in J^+(p)$, then there exists a sequence of points $q_n \in I^+(p)$ with $q_n \xrightarrow{n\to\infty} q$, i.e.

$$J^+(p) \subset \operatorname{clos}\big(I^+(p)\big).$$

Hint: Starting from a causal curve γ connecting p to q, you need to find a sequence of timelike curves γ_n emanating from p converging to γ. To this end, if T is a globally timelike vector field on $\mathcal{M}$, consider variations $\gamma_s(t)$ of $\gamma(t) = \gamma_0(t)$ such that the variation vector field $\frac{\partial}{\partial s}(\gamma_s(t))|_{s=0}$ is of the form $f(t)T|_{\gamma(t)}$ for an appropriately chosen function f.

(b) Assume, moreover, that $(\mathcal{M}, g)$ is *globally hyperbolic*. Prove that, in this case

$$J^+(p) = \operatorname{clos}\big(I^+(p)\big).$$

(c) Can you find an example of a (necessarily not globally hyperbolic) spacetime $(\mathcal{M}, g)$ with a point $p \in \mathcal{M}$ such that $J^+(p)$ is not closed?

7.3 In this exercise, we will explore some of the geometric properties of the Riemann curvature tensor. To this end, let us fix a smooth Lorentzian manifold $(\mathcal{M}, g)$. Recall that

$$R(X,Y)Z \doteq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

(A) Let $\phi : (-\epsilon, \epsilon) \times [0,1] \to \mathcal{M}$ be a smooth map such that, for each $s \in (-\epsilon, \epsilon)$, $\gamma_s = \phi(s, \cdot)$ is a *geodesic*. Define the vector fields $T = d\phi(\frac{\partial}{\partial t})$ and $X = d\phi(\frac{\partial}{\partial s})$.

(a) Prove that $[T, X] = 0$. *(Hint: Compare $\nabla_X T$ and $\nabla_T X$.)*

(b) Let us define the acceleration vector field

$$a = \nabla_T \nabla_T X.$$

Prove that

$$a = -R(X,T)T.$$

Intuitively, X measures the infinitesimal separation between nearby geodesics; thus, when the right hand side above is non-zero, nearby geodesics tend to accelerate towards or away from each other.

(B) Let $\gamma : [0,1] \to \mathcal{M}$ be a smooth curve. For any $t_1, t_2 \in [0,1]$, we will denote with $\mathbb{P}_{\gamma(t_1)\to\gamma(t_2)} : T_{\gamma(t_1)}\mathcal{M} \to T_{\gamma(t_2)}\mathcal{M}$ the parallel transport along γ from $\gamma(t_1)$ to $\gamma(t_2)$.

(a) Prove that, for any vector field Z along γ, as $\tau \to 0$:

$$\lim_{\tau\to 0} \frac{Z|_{t=0} - \mathbb{P}_{\gamma(\tau)\to\gamma(0)} Z|_{t=\tau}}{\tau} = -\nabla_{\dot\gamma(0)} Z.$$

Hint: Contruct a frame $\{e_i\}_{i=1}^n$ of vector fields along γ which are parallel translated, and express Z in components with respect to e_i.

*(b) Let $\phi : [-1,1]\times[-1,1] \to \mathcal{M}$ be a smooth map with $p = \phi(0,0)$ and let $X = \phi^*(\frac{\partial}{\partial x^1})$ and $Y = \phi^*(\frac{\partial}{\partial x^2})$. For any $s_1, s_2 \in (0,1)$, we will consider the rectangular loop $\gamma_{(s_1,s_2)}$ starting and ending at p which is of the form $\gamma_{(s_1,s_2)} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\begin{aligned}
\gamma_1(t) &= \phi(t,0), \quad t \in [0,s_1],\\
\gamma_2(s) &= \phi(s_1,s), \quad s \in [0,s_2],\\
\gamma_3(t) &= \phi(s_1 - t, s_2), \quad t \in [0,s_1],\\
\gamma_3(s) &= \phi(0, s_2 - s), \quad s \in [0,s_2].
\end{aligned}$$

For any $Z \in T_p\mathcal{M}$, let $Z_{(s_1,s_2)} \in T_p\mathcal{M}$ be the tangent vector obtained after parallel transporting Z_p around γ, i.e. following the successive mappings

$$\begin{aligned}
Z \to & Z' = \mathbb{P}_{\gamma_1(0)\to\gamma_1(s_1)} Z \to Z'' = \mathbb{P}_{\gamma_2(0)\to\gamma_2(s_2)} Z'\\
&\to Z''' = \mathbb{P}_{\gamma_3(0)\to\gamma_3(s_1)} Z'' \to Z_{(s_1,s_2)} = \mathbb{P}_{\gamma_4(0)\to\gamma_4(s_2)} Z'''.
\end{aligned}$$

Show that

$$\lim_{s_2\to 0}\lim_{s_1\to 0} \frac{Z_{(s_1,s_2)} - Z}{s_1 s_2} = -R(X,Y)Z.$$

7.4 Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold of dimension $n+1$ and let R be its Riemann curvature tensor.

(a) Show that, in any local coordinate chart $(x^0, \dots, x^n)$ on $\mathcal{M}$, the components of R satisfy the following identities:

1. $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$.
2. $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$.
3. $R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$ (*First Bianchi identity*).
4. $\nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} = 0$ (*Second Bianchi identity*).

Prove that the Ricci tensor satisfies:

$$g^{\alpha\beta}\nabla_\alpha\big(Ric_{\beta\gamma} - \frac{1}{2}Rg_{\beta\gamma}\big) = 0.$$

That is to say, the Einstein tensor of every Lorentzian manifold is *divergence free*.

(b) Prove that, when $n+1 = 3$, $R_{\alpha\beta\gamma\delta}$ has exactly 6 independent components. Noting that this is the same number of independent components as for the Ricci tensor $Ric_{ij} = g^{ab}R_{iajb}$, can you prove that, when $n+1 = 3$, $Ric_{ij} = 0$ implies that $R_{ijkl} = 0$? How many independent components do these tensors have in dimension $n+1 = 2$?

(c) When $n + 1 \geqslant 3$, define the Weyl tensor by the relation

$$\begin{aligned} W_{\alpha\beta\gamma\delta} = {} & R_{\alpha\beta\gamma\delta} + \frac{1}{n-1}\Big(Ric_{\alpha\delta}g_{\beta\gamma} - Ric_{\alpha\gamma}g_{\beta\delta} + Ric_{\beta\gamma}g_{\alpha\delta} - Ric_{\beta\delta}g_{\alpha\gamma}\Big) \\ & + \frac{1}{n(n-1)}R\big(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\big) \end{aligned}$$

where $R = g^{ij}Ric_{ij}$ is the Ricci scalar. Prove that the Weyl tensor satisfies the same symmetries as the Riemann tensor, and moreover

$$g^{\alpha\gamma}W_{\alpha\beta\gamma\delta} = 0.$$

Deduce that $W_{\alpha\beta\gamma\delta} = 0$ when $n + 1 = 3$.

*(d) Let $\phi : \mathcal{M} \to \mathbb{R}_+$ be a C^∞ function and consider the conformal metric

$$\tilde{g} = \phi^2 g.$$

Show that the Weyl tensors of g and $\tilde{g}$ satisfy

$$\tilde{W}^\alpha{}_{\beta\gamma\delta} = W^\alpha{}_{\beta\gamma\delta},$$

i.e. W is a *conformal invariant* of g. Deduce that a necessary condition for a metric g to be conformally flat, i.e. of the form $\phi^2\eta$, is that $W = 0$ (it can be shown that it is also a sufficient condition when $\dim\mathcal{M} > 3$).